About estimations of difference for the partial integro-differential equation with small parameter at the leading derivative

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Abstract

This paper is devoted to the study of the singularly perturbed second order partial integro-differential equations. The estimation of the solutions of Cauchy problem is obtained.

1 Introduction

Constructing of asymptotic decompositions of solutions of singularly perturbed differential equations has great theoretical and practical importance. In this line of investigations, the fundamental results obtained by A.N. Tihonov [1], A.B. Vasyliyeva [2], K.A. Kasymov [3], M.I. Imanaliyev [4], L.A. Lyusternik [5] and others.

However, the results, obtained in [1]-[5], still not generalized for the partial second order integro-differential equations of Volterra type.

The aim of this paper is the study of the Cauchy problem with initial jump for the singularly perturbed partial second order integro-differential equations.

2 Preliminaries

Consider in $G = \{(t,x) : 0 \leq t \leq 1, \lambda \leq x \leq \lambda + 1\}$ following problem

$$L_\varepsilon y = \varepsilon H^2[y] + A(t,x)H[y] + B(t,x)y =$$

$$= F(t,x) + \int_0^t (K_1(t,s,x)H[y(s,x)] + K_0(t,s,x)y(s,x))\,ds, \tag{1}$$

$$y(0,x,\varepsilon) = \pi_0(x), \qquad \varepsilon \cdot y_t^{'}(0,x,\varepsilon) = \pi_1(x). \tag{2}$$

Here $\varepsilon > 0$ - a small parameter, t, x - independent variables, $y = y(t,x,\varepsilon)$ - unknown function, $A(t,x)$, $B(t,x)$, $F(t,x)$, $K_i(t,s,x)$ and $\pi_i, (i = 0,1)$ - functions given in G, operators

$$H[y] =< e(t,x) \cdot grady >, \quad H^2[y] = H[H[y]],$$

where $< \cdot >$ denotes inner product of vectors $e(t,x) = (1, Q(t,x))$ and $grady = \left(\frac{\partial y}{\partial t}, \frac{\partial y}{\partial x}\right)$, $Q(t,x)$ is also given in G, the function $\lambda(t)$ is a solution of characteristic equation

$$\frac{dx}{dt} = Q(t,x). \tag{3}$$

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Consider also disturbed problem

$$L_0 y_0 = A(t,x)H[y_0] + B(t,x)y_0 =$$

$$= F(t,x) + \int_0^t (K_1(t,s,x)H[y_0(s,x)] + K_0(t,s,x)y_0(s,x))\,ds, \tag{4}$$

$$y_0(0,x) = \pi_0(x). \tag{5}$$

obtained from (1), (2) when $\varepsilon = 0$.

Suppose, that

1) $A(t,x)$, $B(t,x)$, $F(t,x)$, $K_i(t,s,x)$, $\lambda(t)$ and $\pi_i, (i = 0,1)$ - continuous functions in G.

2) conditions

$$\inf_{(t,x)\in G} A(t,x) \geq \gamma > 0, \qquad \inf_{(t,x)\in G} Q(t,x) \geq \sigma > 0, \qquad \inf_{(t,x)\in G} \pi_i(x) \geq \sigma > 0,$$

$$\lambda(0) = 0, \lambda(1) = 1,$$

is satisfied, where γ and σ - some real numbers.

3 Estimation of difference

In this section, I prove estimations of the difference between perturbed and unperturbed Cauchy problems.

Using theorem 2 from [6], is not difficult to prove, that solution $y(t,x,\varepsilon)$ of problem (1), (2) doesn't goes to solution of problem (4), (5) when $\varepsilon \to 0$.

Consider the following problem

$$L_0 y_0 = F(t,x) + \int_0^t (K_1(t,s,x)H[y_0(s,x)] + K_0(t,s,x)y_0(s,x))\,ds + \Delta(t,x), \tag{6}$$

$$y_0(0,x) = \pi_0(x) + \Delta_0(x), \tag{7}$$

where $\Delta(t,x)$, $\Delta_0(x)$ - not for a while yet unknown functions.

Function $\Delta_0(x)$ is call to be named initial jump of solution of problem (1), (2), function $\Delta(t,x)$ - initial jump of integral term of equation (1).

Suppose, that solution $y_0(t,x)$ of problem (6), (7) when $t = t_0 = \frac{\varepsilon}{\gamma}|\ln\varepsilon|$ satisfies the condition

$$y_0(t_0,x) = y(t_0,x,\varepsilon), \quad x \in G. \tag{8}$$

Theorem 1. *Let conditions 1),2) be satisfied. Then for the difference between solution of problem (1), (2) and the solution of problem (6), (7) in $G_1 \subset G$ have place following estimations*

$$|y(t,x,\varepsilon) - y_0(t,x)| \leq K\cdot\varepsilon\cdot|\ln\varepsilon| + K\cdot\varepsilon\cdot|y_t^{'}(t_0,x,\varepsilon)| + K\cdot \max_{(t,x\in G)} |K_1(t,0,x)\cdot\Delta_0(x) - \Delta(t,x)|,$$

$$|y_t^{'}(t,x,\varepsilon) - y_{0t}^{'}(t,x)| \leq K\cdot\varepsilon\cdot|\ln\varepsilon| + K\cdot\varepsilon\cdot|y_t^{'}(t_0,x,\varepsilon)| +$$

$$+ K\cdot \max_{(t,x\in G)} |K_1(t,0,x)\cdot\Delta_0(x) - \Delta(t,x)| + K\cdot\Big(1 + |y_t^{'}(t_0,x,\varepsilon)|\Big)\cdot e^{-\frac{\gamma}{\varepsilon}(t-t_0)},$$

$$|y_x^{'}(t,x,\varepsilon) - y_{0x}^{'}(t,x)| \leq K\cdot\varepsilon\cdot|\ln\varepsilon| + K\cdot\varepsilon\cdot|y_t^{'}(t_0,x,\varepsilon)| +$$

$$+ K\cdot \max_{(t,x\in G)} |K_1(t,0,x)\cdot\Delta_0(x) - \Delta(t,x)| + K\cdot\Big(1 + |y_t^{'}(t_0,x,\varepsilon)|\Big)\cdot e^{-\frac{\gamma}{\varepsilon}(t-t_0)}, \tag{9}$$

where K - some constant independent on t and ε, $G_1 = \{(t,x) : 0 < t_0 \leq t \leq 1, \lambda \leq x \leq \lambda + 1\}$.

Proof. Indeed, in (1) assign $y(t,x,\varepsilon) = y_0(t,x) + u(t,x,\varepsilon)$, and taking into consideration (7), (8), we obtain for $u(t,x,\varepsilon)$ following problem

$$L_\varepsilon u = f(t,x,\varepsilon) + \int_{t_0}^{t} (K_1(t,s,x)H[u(s,x,\varepsilon)] + K_0(t,s,x)u(s,x,\varepsilon))\,ds, \tag{10}$$

$$u(t_0,x,\varepsilon) = 0, \qquad u_t^{'}(t_0,x,\varepsilon) = y_t^{'}(t_0,x,\varepsilon), \tag{11}$$

where function $f(t,x,\varepsilon)$ has a representation

$$f(t,x,\varepsilon) = K_1(t,0,\varphi)\cdot\Delta_0(\psi) + \int_{t_0}^{t}\left(K_0(t,s,\varphi) - \frac{\partial K_1(t,s,\varphi)}{\partial s}\right)u(s,\varphi,\varepsilon)ds - \Delta(t,x) - \varepsilon H^2[y_0(t,x)],$$

and estimation

$$|f(t,x,\varepsilon)| \le K\cdot\varepsilon\cdot|\ln\varepsilon| + \max_{(t,x\in G)}|K_1(t,0,x)\cdot\Delta_0(x) - \Delta(t,x)|, \quad (t,x)\in G, \tag{12}$$

where $\varphi = \varphi(t,\psi)$ - a solution of characteristic equation (3), $\psi = \psi(t,x) = x_0$ - first integral of equation (1) [6]. Applying to the problem (10), (11) theorem 1 from [6], and taking into consideration (12), we obtain (9). Theorem is proved.

4 Initial jumps of solutions and integral term

The aim of this section is to define the conditions in the presence of which the solution of perturbed Cauchy problem goes to the solution of unperturbed problem.

Taking into consideration (9), assign

$$\Delta(t,x) = \Delta_0(x)\cdot K_1(t,0,x). \tag{13}$$

Then from theorem1, we obtain, that

$$\lim_{x\to 0} y(t,x,\varepsilon) = y_0(t,x), \quad \lim_{x\to 0} y_t^{'}(t,x,\varepsilon) = \frac{\partial y_0(t,x)}{\partial t}, \quad (t,x)\in G_1.$$

Further, for define $\Delta_0(x)$, integrate equation (1) along characteristic $x = \varphi(t,\psi)$ on t from 0 to t_0. Then we have

$$\varepsilon\cdot H[y_0(t_0,\varphi,\varepsilon)] - \pi_1(\psi) + A(t_0,\varphi)y_0(t_0,\varphi,\varepsilon) - A(0,\varphi)\pi_0(\psi) - \int_0^{t_0}\left(A_t^{'}(t,\varphi) - B(t,\varphi)\right)\times$$

$$\times y(t,\varphi,\varepsilon)dt = \int_0^{t_0}\left(F(t,\varphi) + \int_0^{t}(K_1(t,s,\varphi)H[y(s,\varphi,\varepsilon)] + K_0(t,s,\varphi)y(s,\varphi,\varepsilon))\,ds\right)dt \tag{14}$$

From (14), passage to the limit when $\varepsilon\to 0$, and taking into consideration (7), (9), (13), obtain

$$\Delta_0(x) = \frac{\pi_1(\psi)}{A(0,\psi)}, \quad \Delta(t,x) = \frac{\pi_1(\psi)}{A(0,\psi)}\cdot K_1(t,0,\varphi). \tag{15}$$

Thus, in G_1, if equalities (15) is satisfied, then difference between solution $y(t,x,\varepsilon)$ of problem (1), (2) and solution $y_0(t,x)$ of problem (6), (7) will be enough small with ε.

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